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DETERMINATION OF THE STATE OF STRESS AND STRAIN OF MULTICONNECTED TRANSTROPIC PLATES

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The solution of problems concerning the state of stress of multiconnected transtropic plates of arbitrary thickness under symmetric and skew-symmetric loadings is constructed in a three-dimensional formulation. As in [1, 2], the semi-inverse method of Vorovich (see [5 — 8], etc.) is used to obtain homogeneous solutions of the Lur'e-Lekhnitskii type [3, 4]. The state of stress of transtropic plates is determined by the method of reducing the problem to the solution of functional systems, described in [9 — 12] in application to isotropic thick plates.

Analogous problems for simply-connected plates have been analyzed by an asymptotic method in [1, 2].

1. Let us consider an elastic homogeneous layer of thickness $2h$ weakened by arbitrarily arranged circular cylindrical cavities whose generators are normal to the flat faces. Let us consider the layer to experience small deformations under the effect of external forces applied to the side surfaces of the cavities Ω_j $(j = 1, 2, \ldots, s)$. The structure of the body material is such that all the directions in planes parallel to the middle plane are equivalent in the sense of the elastic properties. We call such materials transtropic [13]. Among them, for example, are "star plastics", DSP-G wood plastics, F-60 veneer [13], cadmium, magnesium, zinc crystals [14, 15], etc.

The equations of the generalized Hooke's law for such materials are [16]

$$\varepsilon_x = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) - \frac{\nu_z}{E_z}\sigma_{zz}, \quad \gamma_{xy} = \frac{1}{G}\sigma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) - \frac{\nu_z}{E_z}\sigma_{zz}, \quad \gamma_{yz} = \frac{1}{G_z}\sigma_{yz}$$

$$\varepsilon_z = -\frac{\nu_z}{E_z}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{E_z}\sigma_{zz}, \quad \gamma_{xz} = \frac{1}{G_z}\sigma_{xz}$$

Let us introduce the dimensionless quantities

$$\xi = \frac{x}{R}, \quad \eta = \frac{y}{R}, \zeta = \frac{1}{\lambda}\frac{z}{R}, \quad \sigma_{ij} = \frac{1}{2G}\sigma_{kl}, \quad \lambda = \frac{h}{R}$$

$$u_i = u_k/R \quad (i, j = \xi, \eta, \zeta;\ k, l = x, y, z)$$

where the variables ξ, η are related to the middle plane of the plate, and R is the radius of one of the cavities. Then the generalized Hooke's law equation can be written in the following form (the prime denotes the derivative with respect to ζ):

$$\begin{aligned}
\sigma_{\xi\xi} &= A_{11}\partial_1 u_\xi + A_{12}\partial_2 u_\eta + \lambda^{-1}A_{13}u_\zeta', \quad \sigma_{\xi\eta} = A_{66}(\partial_2 u_\xi + \partial_1 u_\eta) \\
\sigma_{\eta\eta} &= A_{12}\partial_1 u_\xi + A_{11}\partial_2 u_\eta + \lambda^{-1}A_{13}u_\zeta', \quad \sigma_{\xi\zeta} = A_{44}(\partial_1 u_\zeta + \lambda^{-1}u_\xi') \\
\sigma_{\zeta\zeta} &= A_{13}(\partial_1 u_\xi + \partial_2 u_\eta) + \lambda^{-1}A_{33}u_\zeta', \quad \sigma_{\eta\zeta} = A_{44}(\partial_2 u_\zeta + \lambda^{-1}u_\eta')
\end{aligned} \tag{1.1}$$

Here

$$\partial_1 = \frac{\partial}{\partial\xi}, \quad \partial_2 = \frac{\partial}{\partial\eta}, \quad A_{33} = \frac{\mu_2}{2}, \quad A_{44} = \frac{1}{2s_0^2}, \quad A_{66} = \frac{1}{2}, \quad A_{13} = \mu_1\nu_z$$

$$A_{11} = \mu_0^{-1}(1 - \nu_2\nu_z), \quad A_{12} = \mu_0^{-1}(\nu + \nu_2\nu_z), \quad \mu_0 = 1 - \nu - 2\nu_2\nu_z, \quad s_0^2 = G / G_z$$

$$\mu_3 = 2\mu_1\nu_z + s_0^{-2}, \quad \mu_2 = 2\mu_1(1 - \nu)\nu_z / \nu_2, \quad \mu_1 = \mu_0^{-1}(1 + \nu), \quad \nu_2 = \nu_z E / E_z, \quad \mu = (1 - 2\nu)^{-1}$$

The strain energy should be positive, hence, the constraints [15]

$$A_{44} > 0, \quad A_{11} > |A_{12}|, \quad (A_{11} + A_{12}) A_{33} > 2A_{13}^2$$

are imposed on the coefficients A_{ij}.

Let us substitute (1.1) into the equilibrium equation. Consequently, we obtain the elasticity theory equation for a transtropic medium in terms of displacements ($D^2 = \partial_1^2 + \partial_2^2$ is the two-dimensional Laplace operator)

$$\begin{aligned} &(\lambda s_0)^{-2}u_\xi'' + D^2u_\xi + \mu_1\partial_1(\partial_1u_\xi + \partial_2u_\eta) + \lambda^{-1}\mu_3\partial_1u_\zeta' = 0 \\ &(\lambda s_0)^{-2}u_\eta'' + D^2u_\eta + \mu_1\partial_2(\partial_1u_\xi + \partial_2u_\eta) + \lambda^{-1}\mu_3\partial_2u_\zeta' = 0 \\ &\lambda^{-2}\mu_2u_\zeta'' + s_0^{-2}D^2u_\zeta + \lambda^{-1}\mu_3(\partial_1u_\xi' + \partial_2u_\eta') = 0 \end{aligned} \qquad (1.2)$$

Now, the boundary value problem can be formulated thus. Find the solution of the system (1.2) satisfying the following boundary conditions:

$$\sigma_{\xi\zeta} = \sigma_{\eta\zeta} = \sigma_{\zeta\zeta} = 0, \quad \zeta = \pm 1 \qquad (1.3)$$

$$\sigma_{rr} = P_r^j(\theta_j, \zeta), \quad \sigma_{r\theta} = P_\theta^j, \quad \sigma_{r\zeta} = P_\zeta^j \quad \text{on } \Omega_j \qquad (1.4)$$

Here r_j, θ_j, ζ is a cylindrical coordinate system coupled to the center of the j-th cavity, P_k^j $(k = r, \theta, \zeta)$ are given external loads which can always be decomposed into symmetric and skew-symmetric components. As in [3], in the tension-compression problem P_r^j, P_θ^j are even but P_ζ^j is an even function of ζ and, conversely, in the bending problem P_r^j, P_θ^j are odd and P_ζ^j is even.

Let us construct the solution of the problems mentioned as a sum of biharmonic, vortex, and potential states of stress by using the method of homogeneous solutions [1 – 3, 5, 6]

$$u_i = u_{iB} + u_{iR} + u_{iP}, \quad \sigma_{ij} = \sigma_{ijB} + \sigma_{ijR} + \sigma_{ijP} \quad (i, \ j = \xi, \eta, \zeta) \qquad (1.5)$$

2. Let us seek the vortex solution in the form

$$u_{\xi R}(\xi, \eta, \zeta) = p(\zeta)\,\partial_2B(\xi, \eta), \quad u_{\eta R} = -p\partial_1B, \quad u_{\zeta R} = 0 \qquad (2.1)$$

It is assumed here that the displacements u_ξ, u_η are projections of the rotor of some function on the ξ, η axes [1, 2, 5, 6].

From the system (1.2) we have

$$\partial_i(\lambda^{-2}s_0^{-2}p''B + pD^2B) = 0 \qquad (i = 1, 2) \qquad (2.2)$$

For (2.1) to satisfy the system (1.2), it is sufficient to require that the expression in parentheses in (2.2) be zero. In conformity with the method of separation of variables, this requirement can be written thus (δ is the separation constant):

$$p''(\zeta) + \delta^2 s_0^2 p(\zeta) = 0 \tag{2.3}$$

$$D^2 B(\xi, \eta) - (\delta / \lambda)^2 B(\xi, \eta) = 0 \tag{2.4}$$

Let us require compliance with the boundary conditions on the flat faces (1.3). Consequently, we obtain

$$p'(\pm 1)\, \partial_i B(\xi, \eta) = 0 \qquad (i = 1, 2)$$

But $\partial_i B \not\equiv 0$ (in the opposite case, we would have a trivial solution), hence

$$p'(\pm 1) = 0 \tag{2.5}$$

Therefore, in connection with finding the function $p(\zeta)$ we arrive at the Sturm-Liouville problem (2.3), (2.5).

Since the displacements u_ξ, u_η are even in the tension-compression problem, but odd functions of the variable ζ in the bending problem, the solution of the problem (2.3), (2.5) can be written as

$$\begin{gathered} p_k^+(\zeta) = b_k^+ \cos \delta_k^+ s_0 \zeta, \quad p_k^-(\zeta) = b_k^- \sin \delta_k^- s_0 \zeta \\ \delta_k^+ = \delta^+ = k\pi / s_0, \quad \delta_k^- = \delta^- = (2k-1)\pi / 2s_0, \quad k = \pm 1, \pm 2, \ldots \\ (\sin \delta^+ s_0 = 0, \quad \cos \delta^- s_0 = 0, \quad \delta^\pm \neq 0) \end{gathered} \tag{2.6}$$

Here $b_k^\pm$ are arbitrary constants and $\delta_k^\pm$ are the roots of the equations in parentheses. The solutions of (2.4) correspond to these same values of $\delta_k^\pm$

It follows from (2.4) and (2.6) that the functions $B_k(\xi, \eta)$ are even and $p_k^\pm(\zeta)$ can be selected because of the corresponding selection of the constants $b_k^\pm$. Hence, summation over $\delta_k^\pm < 0$ does not yield new solutions, and we can finally write for both problems

$$u_{\xi R}(\xi, \eta, \zeta) = \sum_{k=1}^{\infty} p_k(\zeta)\, \partial_2 B_k(\xi, \eta), \quad u_{\eta R} = -\sum_{k=1}^{\infty} p_k \partial_1 B_k, \quad u_{\zeta R} = 0 \tag{2.7}$$

Substituting (2.7) into (1.1), we obtain

$$\sigma_{\xi\xi R} = -\sigma_{\eta\eta R} = \sum_{k=1}^{\infty} p_k \partial_1 \partial_2 B_k, \quad \sigma_{\xi\eta R} = \frac{1}{2} \sum_{k=1}^{\infty} p_k (\partial_2^2 - \partial_1^2) B_k \tag{2.8}$$

$$\sigma_{\xi\zeta R} = -\sum_{k=1}^{\infty} g_k(\zeta)\, \partial_2 B_k, \quad \sigma_{\eta\zeta R} = \sum_{k=1}^{\infty} g_k \partial_1 B_k, \quad \sigma_{\zeta\zeta R} = 0$$

The notation in (2.7) and (2.8) is

$$\begin{gathered} p_k^+(\zeta) = 2\lambda s_0 \cos \delta_k^+ s_0 \zeta, \quad p_k^-(\zeta) = \frac{2\lambda s_0}{\delta_k^-} \sin \delta_k^- s_0 \zeta \\ g_k^+(\zeta) = \delta_k^+ \sin \delta_k^+ s_0 \zeta, \quad g_k^-(\zeta) = -\cos \delta_k^- s_0 \zeta \end{gathered} \tag{2.9}$$

In the isotropic case $G_z = G$, and therefore, $s_0 = 1$. In this case (2.7) – (2.9) agree with those presented in [3].

3. Let us seek the potential solution in the form [5, 6]

$$u_{\xi P}(\xi, \eta, \zeta) = n(s)\, \partial_1 C(\xi, \eta), \quad u_{\eta P} = n \partial_2 C, \quad u_{\zeta P} = q(\zeta)\, C \tag{3.1}$$

The functions $n(\zeta)$, $q(\zeta)$, $C(\xi, \eta)$ are determined, as in Sect. 2, by satisfying (1.2) and the conditions (1.3).

From the system (1.2) we obtain

$$\partial_i\,[(\lambda s_0)^{-2}n''(\zeta) + (1+\mu_1)\,D^2Cn(\zeta) + \lambda^{-1}\mu_3 q'(\zeta)\,C] = 0 \qquad (3.2)$$
$$\lambda^{-2}\mu_2 q''(\zeta)\,C + (s_0^{-2}q + \lambda^{-1}\mu_3 n')\,D^2C = 0 \quad (i=1,2)$$

The variables separate in (3.2) if we set

$$D^2C = (\gamma/\lambda)^2 C \qquad (3.3)$$

Taking account of the dependences (3.2) and (3.3), we can satisfy the system (1.2) by requiring compliance with the following conditions

$$n''(\zeta) + (1+\mu_1)\gamma^2 s_0^2 n(\zeta) + \lambda\mu_3 s_0^2 q'(\zeta) = 0 \qquad (3.4)$$
$$q''(\zeta) + (\gamma s_0)^2\mu_2^{-1} q(\zeta) + \gamma^2\mu_3(\lambda\mu_2)^{-1} n'(\zeta) = 0$$

We seek the solution of the system (3.4) by the Euler method. Its characteristic equation is

$$S^4 + 2b_1\gamma^2 S^2 + b_2\gamma^4 = 0, \quad b_1 = \frac{s_0^2 - \nu_2}{1-\nu}, \quad b_2 = \frac{\nu_2}{\nu_z}\frac{1-\nu_2\nu_z}{1-\nu^2} \qquad (3.5)$$

To write the general solution of the system (3.4), let us consider the following possible cases:

1°. If $b_1 > 0$ and $b_1^2 - b_2 \neq 0$, then

$$S_{1,2} = \pm i\gamma s_1, \qquad S_{3,4} = \pm i\gamma s_2 \qquad (3.6)$$
$$n^+(\zeta) = H_1^+\cos\gamma^+ s_1\zeta + H_2^+\cos\gamma^+ s_2\zeta,$$
$$n^-(\zeta) = H_1^-\sin\gamma^- s_1\zeta + H_2^-\sin\gamma^- s_2\zeta$$
$$q^+(\zeta) = Q_1^+\sin\gamma^+ s_1\zeta + Q_2^+\sin\gamma^+ s_2\zeta,$$
$$q^-(\zeta) = Q_1^-\cos\gamma^- s_1\zeta + Q_2^-\cos\gamma^- s_2\zeta$$

Hence

$$s_{1,2} = \sqrt{b_1 \pm \sqrt{b_1^2 - b_2}}$$

are real and different if $b_1^2 - b_2 > 0$ and complex conjugate if $b_1^2 < b_2$.

2°. If $b_1 > 0$ and $b_1^2 = b_2$, then

$$S_{1,2} = S_{3,4} = \pm i\gamma s_1, \quad s_1 = \sqrt{b_1} \qquad (3.7)$$
$$n^+(\zeta) = H_1^+\cos\gamma^+ s_1\zeta + H_2^+\zeta\sin\gamma^+ s_1\zeta, \quad n^- = H_1^-\sin\gamma^- s_1\zeta +$$
$$H_2^-\zeta\cos\gamma^- s_1\zeta$$
$$q^+(\zeta) = Q_1^+\sin\gamma^- s_1\zeta + Q_2^+\zeta\cos\gamma^+ s_1\zeta, \quad q^- = Q_1^-\cos\gamma^- s_1\zeta +$$
$$Q_2^-\zeta\sin\gamma^- s_1\zeta$$

In particular, if $\nu_z = \nu_2 = \nu$ and $G_z = G$, then $s_0^2 = s_1 = b_1 = 1$, i.e. the solution for an isotropic plate is obtained from this case.

3°. If $b_1 < 0$ and $b_1^2 - b_2 \neq 0$, then $S_{1,2} = \pm\gamma s_1$, $S_{3,4} = \pm\gamma s_2$. Hence

$$s_{1,2} = \sqrt{|b_1| \pm \sqrt{b_1^2 - b_2}}$$

when $b_1^2 < b_2$ and

$$s_{1,2} = \sqrt{|b_1| \pm i\sqrt{b_2 - b_1^2}}$$

when $b_1^2 < b_2$. The expression for $n^\pm(\zeta)$ and $q^\pm(\zeta)$ are obtained from (3.6) by replacing the circular by corresponding hyperbolic functions.

4°. If $b_1 < 0$ and $b_1^2 = b_2$, then $S_{1,2} = S_{3,4} = \pm\gamma s_1$, $s_1 = \sqrt{|b_1|}$. The expressions for $n^\pm(\zeta)$ and $q^\pm(\zeta)$ are obtained from (3.7) by using the same substitution as in case 3°.

The constants $H_m^{\pm}$, $Q_m^{\pm}$ $(m = 1, 2)$ in (3.6), (3.7) are expressed in terms of each other. For instance

$$Q_m^+ = A_m^+ H_m^+, \quad Q_m^- = -A_m^- H_m^-, \qquad (3.8)$$
$$A_m^{\pm} = (\gamma^{\pm} / \lambda)\, s_m \mu_3 s_0^2 (1 - s_0^2 \mu_2 s_m^2)^{-1}$$

Analogous relations are established in other cases also.

We determine the constants $H_m^{\pm}$ from the boundary conditions on the flat faces (1.3). For example, for case 1° we obtain

$$a_1 \cos \gamma^+ s_1 H_1^+ + a_2 \cos \gamma^+ s_2 H_2^+ = 0, \quad a_1 \sin \gamma^- s_1 H_1^- + a_2 \sin \gamma^- s_2 H_2^- = 0 \qquad (3.9)$$
$$d_1 s_1 \sin \gamma^+ s_1 H_1^+ + d_2 s_2 \sin \gamma^+ s_2 H_2^+ = 0, \quad d_1 s_1 \cos \gamma^- s_1 H_1^- + d_2 s_2 \cos \gamma^- s_2 H_2^- = 0$$
$$a_m = \left(1 + \frac{1-\nu}{\nu_2} s_m^2\right)(1 - \mu_2 s_0^2 s_m^2)^{-1}, \quad d_m = 2\nu_z \mu_1 s_0^m a_m$$

For a non-trivial solution of the system (3.9) to exist it is necessary that their determinant be zero. Hence, an equation to determine γ follows, which can be written as [4]

$$(s_1 + s_2) \sin (s_1 - s_2) \gamma^{\pm} \pm (s_1 - s_2) \sin \pm (s_1 + s_2) \gamma^{\pm} = 0 \qquad (3.10)$$

Transcendental equations determine the eigenvalues of the appropriate homogeneous problems for the potential state of stress (parameters $\gamma_p^{\pm}$). The eigenfunctions $n_p^{\pm}(\zeta)$ and $q_p^{\pm}(\zeta)$, as well as the functions $C_p(\xi, \eta)$ determined from (3.3) correspond to these eigenvalues.

It follows from (3.9) that

$$H_{2p}^+ = -\frac{a_1 \cos s_1 \gamma_p^+}{a_2 \cos s_2 \gamma_p^+} H_{1p}^+, \quad H_{2p}^- = -\frac{a_1 \sin s_1 \gamma_p^-}{a_2 \sin s_2 \gamma_p^-} H_{1p}^-$$

Equations (1.2) and conditions (1.3) will be satisfied when the constants $H_{1p}^{\pm}$ in (3.6) remain arbitrary.

According to (3.3), the functions $C_p^{\pm}(\xi, \eta)$ are even in $\gamma_p^{\pm}$. Hence, the constants $H_{1p}^{\pm}$ are selected so that the displacements would be even in $\gamma_p^{\pm}$, which permits consideration of just those roots of (3.10) whose real part is greater than zero. We take the constants mentioned as $H_{1p}^+ = \cos \gamma_p^+ s_2$, $H_{1p}^- = \sin \gamma_p^- s_2$. Then

$$n_p^+(\zeta) = \cos \gamma_p^+ s_2 \cos \gamma_p^+ s_1 \zeta - s_3 \cos \gamma_p^+ s_1 \cos \gamma_p^+ s_2 \zeta \qquad (3.11)$$
$$s_3 = a_1 / a_2$$
$$q_p^+(\zeta) = S_{1p}^+ \cos \gamma_p^+ s_2 \sin \gamma_p^+ s_1 \zeta - S_{2p}^+ s_3 \cos \gamma_p^+ s_1 \sin \gamma_p^+ s_2 \zeta,$$
$$S_{mp}^+ = A_m^+(\gamma_p)$$

Expressions for $n_p^-(\zeta)$, $q_p^-(\zeta)$ are derived from those produced by substitution of $\cos x^+$ for $\sin x^-$, and $\sin x^+$ for $-\cos x^-$, where $x^{\pm} = s_m \gamma_p^{\pm}$ or $x^{\pm} = s_m \gamma_p^{\pm} \zeta$.

The formulas for the displacements can now be written thus:

$$u_{\xi P} = \sum_{p=1}^{\infty} n_p(\zeta) \partial_1 C_p(\xi, \eta), \quad u_{\eta P} = \sum_{p=1}^{\infty} n_p \partial_2 C_p, \quad u_{\zeta P} = \sum_{p=1}^{\infty} q_p C_p \qquad (3.12)$$

From Hooke's law equation we have

$$\sigma_{\xi\xi P} = \sum_{p=1}^{\infty} [s_p(\zeta) + n_p(\zeta) \partial_1^2] C_p, \quad \sigma_{\xi\zeta P} = \sum_{p=1}^{\infty} r_p(\zeta) \partial_1 C_p \qquad (3.13)$$

$$\sigma_{\eta\eta P} = \sum_{p=1}^{\infty} [s_p(\zeta) + n_p(\zeta)\partial_2^2] C_p, \quad \sigma_{\eta\zeta P} = \sum_{p=1}^{\infty} r_p(\zeta)\partial_2 C_p$$

$$\sigma_{\zeta\zeta P} = \sum_{p=1}^{\infty} t_p(\zeta) C_p, \quad \sigma_{\xi\eta P} = \sum_{p=1}^{\infty} n_p(\zeta)\partial_1\partial_2 C_p$$

$$s_p(\zeta) = (\gamma_p / \lambda)^2 n_p(\zeta) A_{12} + \lambda^{-1} A_{13} q_p'(\zeta)$$

$$r_p(\zeta) = {}^1/_2 s_0^{-2}(q_p + \lambda^{-1} n_p'), \; t_p(\zeta) = (\gamma_p / \lambda)^2 A_{13} n_p + \lambda^{-1} q_p' A_{33}$$

Let us transform (3.10). If $b_1 > 0$ and $b_1^2 > b_2$, then by using the notation $s_1 + s_2 = \Omega$, $(s_1 - s_2) / (s_1 + s_2) = \omega$ (Ω and ω are real), we obtain

$$\omega \sin \Omega\gamma^\pm \pm \sin \omega\gamma^\pm\Omega = 0 \tag{3.14}$$

If $b_1 > 0$ and $b_1^2 < b_2$, then $s_{1,2} = \alpha \pm i\beta = \sqrt{b_1 \pm i\sqrt{b_2 - b_1^2}}$. We then have [17]

$$\beta \sin 2\alpha\gamma^\pm \pm \alpha \operatorname{sh} 2\beta\gamma^\pm = 0 \tag{3.15}$$

In case 2° the constants $H_m^\pm$ are found in an analogous manner but have a more awkward structure. The characteristic equation is

$$2s_1\gamma^\pm \pm \sin 2s_1\gamma^\pm = 0 \tag{3.16}$$

As regards the cases 3° and 4°, the results for them are obtained from cases 1° and 2° by the formal replacement of s_1, s_2 by is, is_2. For example, the equations to determine $\gamma_p^\pm$ are obtained from (3.14) – (3.16) and are represented thus

$$\omega \operatorname{sh}\Omega\gamma^\pm \pm \operatorname{sh}\omega\Omega\gamma^\pm = 0, \quad \beta\operatorname{sh}2\alpha\gamma^\pm \pm \alpha\sin 2\beta\gamma^\pm = 0, \quad 2s_1\gamma^\pm \pm \operatorname{sh}2s_1\gamma^\pm = 0$$

The stresses and displacements are calculated by means of (3.12) and (3.13) in which the expressions for $n_p(\zeta)$ and $q_p(\zeta)$ have a structure of the form (3.11).

4. Let us seek the displacement vector components of the biharmonic solution as

$$u_{\xi B}^+ = \partial_1(\Phi_0 + \zeta^2\Phi_2 + \Phi_0^*), \quad u_{\eta B}^+ = \partial_2(\Phi_0 + \zeta^2\Phi_2 - \Phi_0^*), \tag{4.1}$$

$$u_{\zeta B}^+ = \zeta\Phi_1$$

$$u_{\xi B}^- = \partial_1(\zeta\Psi_1 + \zeta^3\Psi_3), \quad u_{\eta B}^- = \partial_2(\zeta\Psi_1 + \zeta^3\Psi_3), \quad u_{\zeta B}^- = \Psi_0 + \zeta^2\Psi_2$$

where $\Phi_m = \Phi_m(\xi, \eta)$, $\Psi_m = \Psi_m(\xi, \eta)$ are some arbitrary functions to be determined. Requiring that (4.1) satisfy the system (1.2) and conditions (1.3), we obtain

$$\Phi_1 = 2\lambda\mu_8 D^2\Phi_0, \quad \Phi_2 = -\lambda^2\mu_8 D^2\Phi_0, \quad \partial_1^2\Phi_0^* = -(1+\nu)^{-1}D^2\Phi_0, \quad D^2D^2\Phi_0 = 0$$

$$\partial_2^2\Phi_0^* = (1+\nu)^{-1} D^2\Phi_0, \quad \Psi_0 = -\frac{1}{\lambda}\Psi_1 + 2\mu_5\lambda s_0^2 D^2\Psi_1,$$

$$\Psi_2 = -\lambda\nu_2\mu_5 D^2\Psi_1, \quad D^2D^2\Psi_1 = 0$$

$$\Psi_3 = -\lambda^2\mu_4 D^2\Psi_1, \quad \mu_4 = {}^1/_3\mu_5(2s_0^2 - \nu_2), \quad \mu_5 = {}^1/_2(1-\nu)^{-1},$$

$$\mu_8 = {}^1/_2\nu_2(1+\nu)^{-1}$$

Let us introduce a new biharmonic function F in place of Φ_0 by means of the relationship

$$\Phi_0 = -(F + {}^1/_3\lambda^2\mu_8\ D^2F), \quad D^2D^2F = 0$$

Then the displacements are written thus

$$\begin{aligned}
&u_{\xi B}^+ = -\partial_1[F + \lambda^2({}^1/_3 - \zeta^2)\mu_8 D^2F - \Phi_0^*], \quad \partial_1{}^2\Phi_0^* = (1+\nu)^{-1}D^2F \qquad (4.2)\\
&u_{\eta B}^+ = -\partial_2[F + \lambda^2({}^1/_3 - \zeta^2)\mu_8 D^2F + \Phi_0^*], \quad \partial_2{}^2\Phi_0^* = -(1+\nu)^{-1}D^2F\\
&u_{\xi B}^- = \partial_1[\zeta F - \lambda^2\mu_4\zeta^3D^2F], \quad F = \Psi_1, \quad u_{\zeta B}^+ = -2\lambda\mu_8\zeta D^2F\\
&u_{\eta B}^- = \partial_2[\zeta F - \lambda^2\mu_4\zeta^3D^2F], \quad u_{\zeta B}^- = -\frac{1}{\lambda}F - \lambda\mu_5(\nu_2\zeta^2 - 2s_0{}^2)D^2F
\end{aligned}$$

Substituting the displacements (4.2) into (1.1), we obtain the following formulas to determine the stresses of the biharmonic state:

$$\begin{aligned}
&\sigma_{\xi\xi B}^+ = \partial_2{}^2[F + \lambda^2\mu_8({}^1/_3 - \zeta^2)D^2F], \quad \sigma_{\eta\eta B}^+ = \partial_1{}^2[F + \\
&\quad \lambda^2\mu_8({}^1/_3 - \zeta^2)D^2F]\\
&\sigma_{\xi\eta B}^+ = -\partial_1\partial_2[F + \lambda^2\mu_8({}^1/_3 - \zeta^2)D^2F], \quad \sigma_{\xi\zeta B}^+ = \sigma_{\eta\zeta B}^+ = \\
&\quad \sigma_{\zeta\zeta B}^+ = 0\\
&\sigma_{\xi\xi B}^- = \zeta(\mu_6\partial_1{}^2 + \mu_7\partial_2{}^2)F - \zeta^3\mu_4\lambda^2\partial_1{}^2D^2F, \quad \sigma_{\xi\zeta B}^- = \\
&\quad \lambda\mu_5(1-\zeta^2)\partial_1D^2F\\
&\sigma_{\eta\eta B}^- = \zeta(\mu_7\partial_1{}^2 + \mu_6\partial_2{}^2)F - \zeta^3\mu_4\lambda^2\partial_2{}^2D^2F, \quad \sigma_{\eta\zeta B}^- = \\
&\quad \lambda\mu_5(1-\zeta^2)\partial_2D^2F\\
&\sigma_{\xi\eta B}^- = \partial_1\partial_2(\zeta F - \zeta^3\lambda^2\mu_4D^2F), \quad \sigma_{\zeta\zeta B}^- = 0, \quad \mu_6 = 2\mu_5,\\
&\mu_7 = \mu_6 - 1
\end{aligned}$$

5. The solution of the problem posed in Sect. 1 reduces to finding the functions F, B_k, C_p which will satisfy the system of governing equations

$$D^2D^2F = 0, \quad D^2C_p = (\gamma_p/\lambda)^2C_p, \quad D^2B_k = (\delta_k/\lambda)^2B_k \qquad (5.1)$$

The total order of the system (5.1) is $D^{2(2+p+k)}$, which requires the formulation of $(2+p+k)$ boundary conditions on Ω_j instead of the three conditions (1.4). Hence, let us use the ideas of the Bubnov-Galerkin method to match the boundary conditions to the governing system. To do this, let us require that the residuals of the boundary conditions (1.4) be orthogonal to the complete system of functions $\{\sin\delta_m{}^\pm s_0\zeta, \cos\delta_m{}^\pm s_0\zeta\}$ in the segment $[-1, 1]$. We consequently obtain the system of boundary conditions needed to satisfy the conditions on the side surfaces of the cavities.

We will have on Ω_j in the tension-compression problem

$$\begin{aligned}
&\varphi(t_j) + t_j\overline{\varphi'(t_j)} + \overline{\psi(t_j)} + {}^1/_2\Lambda_{1,j}(B_0, C_p) = {}^1/_2 f_{1,0}(t_j) \qquad (5.2)\\
&\quad 16(\lambda/\delta_m^+s_0)^{-2}\mu_8\overline{\varphi''(t_j)} + \Lambda_{1,j}(B_m, C_p) = f_{1,m}(t_j),\\
&\Lambda_{2,j}(B_m, C_p) = f_{2,m}(t_j)
\end{aligned}$$

$$\Lambda_{1,j}(B_m, C_p) = \int_{s_j}\Big[2s_0\lambda(-1)^m L_{8\Omega_j}B_m + \sum_{p=1}^{\infty}(s_{mp}^+L_{0\Omega_j} + n_{mp}^+L_{9\Omega_j})C_p\Big]R_j\,d\sigma_j$$

$$\Lambda_{2,j}(B_m, C_p) = -\delta_m^+(-1)^m L_{2\Omega_j} B_m + \sum_{p=1}^{\infty} r_{mp}^+ L_{1\Omega_j} C_p \quad (m = 1, 2, \ldots)$$

$$F = \mathrm{Re}\,[\bar{z}\varphi(z) + \chi(z)], \quad \psi = \frac{d\chi}{dz}, \quad B_0 = 0, \quad f_{2,m}^-(t_j) = P_{m\zeta}^j$$

$$f_{1,m} = \int_{s_j} (P_{mr}^j + iP_{m\theta}^j) R_j \, d\sigma_j$$

$$\begin{Bmatrix} s_{mp}^+ \\ P_{mr}^j \end{Bmatrix} = (-1)^m \int_{-1}^{1} \begin{Bmatrix} s_p^+ \\ P_r^j \end{Bmatrix} \cos \delta_m^+ s_0 \zeta \, d\zeta,$$

$$\begin{Bmatrix} r_{mp}^+ \\ P_{m\zeta}^j \end{Bmatrix} = (-1)^m \int_{-1}^{1} \begin{Bmatrix} r_p^+ \\ P_\zeta^j \end{Bmatrix} \sin \delta_m^+ s_0 \zeta \, d\zeta.$$

The arbitrary constant which does not influence the stress distribution is discarded in (5.2), t_j is the affix of a point of the j-th contour, $L_q\Omega_j$ are the boundary values of the operators L_q $(q = 0, 1, \ldots, 9)$, which are presented in [11].

We have correspondingly in the bending problem (D_{1j} are real and D_{2j} are imaginary constants)

$$\varkappa\varphi(t_j) + t_j\overline{\varphi'(t_j)} + \overline{\psi(t_j)} - \varkappa_{2m}\overline{\varphi''(t_j)} - \mathrm{X}_{1,j}(B_m, C_p) -$$
$$iD_{1j}t_j + D_{2j} = -\int_{s_j} \left(P_{mr}^j + iP_{m\theta}^j + i\int_{s_j} P_{m\zeta}^j \, ds_j \right) dt_j$$

$$8\mu_5 \,\mathrm{Im}\,\varphi'(t_j) + \mathrm{X}_{2,j}(B_m, C_p) + D_{1j} = \int_{s_j} P_{m\zeta}^j \, ds_j \quad (m = 1, 2, \ldots)$$

$$\mathrm{X}_{1,j}(B_m, C_p) = \int_{s_j} \left\{ \frac{1}{2}(-1)^{m+1} (\delta_m^- s_0)^2 b_m^- \left[L_{8\Omega_j} + \frac{i}{2}\left(\frac{\delta_m^- s_0}{\lambda}\right)^2 L_{0\Omega_j} \right] \times \right.$$
$$\left. B_m + \sum_{p=1}^{\infty} \left[(s_{mp}^- L_{0\Omega_j} + n_{mp}^- L_{9\Omega_j}) C_p + r_{mp}^- \int_{s_j} iL_{1\Omega_j} C_p \, ds_j \right] \right\} R_j \, d\sigma_j$$

$$\mathrm{X}_{2,j}(B_m, C_p) = (-1)^{m+1} \frac{(\delta_m^- s_0)^4}{4\lambda^2} b_m^- L_{0\Omega_j} B_m + \sum_{p=1}^{\infty} r_{mp}^- \int_{s_j} L_{1\Omega_j} C_p \, ds_j$$

$$\varkappa = -(3 + \nu)/(1 - \nu), \quad \varkappa_{2m} = 12\lambda^2\mu_4 [1 - 2/(\delta_m^- s_0)^2]$$

$$\int_{-1}^{1} \begin{Bmatrix} s_p^- \\ P_r^j \end{Bmatrix} \sin \delta_m^- s_0 \, \zeta d\zeta = \frac{2(-1)^{m+1}}{(\delta_m^- s_0)^2} \begin{Bmatrix} s_{mp}^- \\ P_{mr}^j \end{Bmatrix},$$

$$\int_{-1}^{1} \begin{Bmatrix} r_p^- \\ P_\zeta^j \end{Bmatrix} \cos \delta_m^- s_0 \, \zeta d\zeta = \frac{2\lambda(-1)^m}{(\delta_m^- s_0)^2} \begin{Bmatrix} r_{mp}^- \\ P_{m\zeta}^j \end{Bmatrix}$$

After having determined F, B_k, C_p, the state of stress and strain at an arbitrary point of the plate is found from (1.5).

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